PhysComp96
Full paper
Draft, 14 October 1996

Non-Replicative Fredkin's Rules in Homogeneous Cellular Spaces

Plamen Petrov
Sofia University, FMI

In the fifties John von Neumann intensely studied the problem of self-reproduction of configurations in computationally universal cellular automata as a model of biological evolution. Later, Edward Fredkin demonstrated that trivial reproduction is easy to obtain, if we ignore the constraints of universality.

In the present work we extend the boundaries of classical notion of cellular automaton (CA), presenting the new notion of "cellular automaton on a homogeneous directed graph", and we demonstrate that applying the rules found by Fredkin on some special structures of this kind does not lead to their well-known replicative behavior.

We suggest these phenomena as a cellular automata model of CP-violation.

1 Introduction

1.1 Preface

The cellular automata (CA), with their specific abilities, like homogenous structure, concurrence and uniformity of the local rules activity (a certain analogy of the physical laws) and the complexity of the configurations engendered by them have become lately the starting point for many interdisciplinary areas like the Evolutionary Biology, Genetic Algorithms, Artificial Life, Theory of Chaos, Physics of Computation, etc.

With CA, the "space" is an infinite regular lattice of "cells", where each cell can comprise a limited number of elements, known as "states". Cells that we conditionally consider being "close" to each other and that can "interact" during a "time cycle" (a conditional "time unit"), we call "neighbors". The evolution of automaton is generally specified by a local function (a "rule"), that for a cell juxtaposes for every certain set of neighbor's states a strictly specified state of this very cell in the following moment (cycle). It is worth noting that all changes in all cells are done simultaneously, i.e. during the same cycle. Another, wider and more formal definition of the term "cellular automaton" will be given later in §2.

May be the most eminent CA is the John Conway's "Game of Life"[1]. In this game the cells cover completely the two-dimensional surface in the form of an infinite checkerboard. Each cell has nine neighbors: the immediate adjacent cells to it—in horizontal, vertical and diagonal direction, and the cell itself. The number of possible states for each cells is limited to two: 0 or 1. The rule of the "Life" game is very simple: if exactly three neighboring cells contain 1, then the central cell becomes 1, if exactly four neighboring cells contain 1, then the "center" remains unchanged. In all other cases the central cell becomes zero with the following cycle.

In earlier works, like those of von Neumann[2], there are examples of two-dimensional automata with an unusual large number of cell states—29 (twenty nine).

In 1983–84 Stephen Wolfram proposes the one-dimensional (known also as "linear") cellular automata [3], [4]. Many researchers consider those works "responsible" for the revival of the interest towards the CA after the initial boom at the end of the Sixties and the beginning of the Seventies.

There are other CA with considerably more complex structures (hexagonal, triangular, cubical, etc.).

1.2 Historical notes

In the Fifties, John von Neumann, after the strong impression of the works of the Polish mathematician Stanislaw Ulam, takes up the problem of the non-trivial reproduction of self-like structures, using the cellular automata for a mathematical model. Von Neumann was interested mainly in the question whether life could be generated by relatively simple mathematical constructions, and to what extent those parallel computation models are equivalent to the computing machine, described by Turing (1936). Unfortunately, the capital work of von Neumann [2] was not finished because of his death in 1957. This classical work, however, finds many admirers and imitators, especially in the Sixties and the Seventies, as well as nowadays [6], [7].

A matter of special interest in von Neumann's work is his definition of the term "non-trivial reproduction". According to his understanding, to be non-trivially reproducing itself a unit should be simultaneously a universal computer and an universal constructor over a set of conditions. The latter requires the possibility for a given imaginative being, that "dwells" similar cellular space, to be able to reproduce itself without comprehensive description of its own being.

While requiring universal computability and constructability, von Neumann does not state at all that these criteria are *necessary*, but that they are *sufficient* for the elimination of the information units that are trivially copying themselves. Therefore, the von Neumann's criterion does not exclude the possibility of the existence of non-universal structures (in any of the above mentioned senses), that can reproduce themselves non-trivially, neither he states that there is any need to require universality for the creatures of nature, so that we could give them the status of "non-trivially reproducing themselves" [8].

Soon after that Edward Fredkin demonstrates that the trivial copying of configurations in CA while neglecting the constraints of computation universality is easily achieved. His solution is based on the rules that set the next state of each cell as a sum of the states of its neighbors modulo k:

$$c\,[t+1] \;:=\; \left(\sum_{i=1}^{n} c_i\,[t]\right) \bmod k \tag{1}$$

where c_i, $i = 1, ..., n$, are the neighbors of cell c, t is time, and k—the number of the states of the cells of the automaton.

Those rules are also known as "parity rules" or "XOR rules", because with binary CA (i.e. when $k = 2$) they take the following form:

> "In case the sum of the sates of the neighboring cells is an even number, then the state of the cell becomes 0 (zero) for the following cycle, otherwise (if it is an odd number), the state of the cell becomes 1 (one)".

The above can be expressed also as:

$$c\,[t+1] := c_1\,[t] \text{ xor } c_2\,[t] \text{ xor } \ldots \text{ xor } c_n\,[t]$$

or:

$$c\,[t+1] := \bigoplus_{i=1}^{n} c_i$$

According to his own words [5], Fredkin comes to this idea back in 1962, but it was first expressed by Martin Gardner in his well known publications, especially those dedicated to the game of "Life" in the "Scientific American" magazine in early Seventies. Those articles, as Gardner says "provoked greater interest among the computer programmers than anything I have ever published", were combined in a separate book[1]. The very three last chapters of this book are the place where the reader meets the Fredkin's elegant example of the trivial reproduction of configurations, known as the "Fredkin's replicator". The most interesting thing in this example is not the very fact of reproduction and not even the beauty of some of the emerging configurations, but the fact that *every*(!) initial configuration under these rules has the ability to reproduce itself in the space, copying itself unlimited number of times (see Fig. 1, 2).

Later many researchers "rediscover" again and again those really surprisingly simple rules and properties and tried in a way to bring in some asymmetry or chaos in this example, with their attempts usually based on the idea of changing the boundary conditions of the structures they investigated. But, as far as we know, such experiments always failed: due to their absolute balance, the Fredkin's rules manifest amazing tenacity in their symmetrical "replicative" behaviour.

Later Fredkin's works are based on combination of XOR rules with other rules in form of the so called "second order CA". Here the cell state in the following time cycle depends both on its neighbors states in the current as well as in the previous cycle of the evolution of the configuration. Those explorations are very popular in "Physics of computation"—a new and perspective interdisciplinary area on the rim between physics and informatics. The basic idea here lies not particularly in the imitation of the process of the biological evolution, but in the modeling of basic physics ideas and laws using reversible CA—automata, where each configuration has no more than one predecessor. The latter is in direct accord with the Second law of thermodynamics and microscopic reversibility, which we observe in our physical world.

1.3 Aims and tasks of the current work

It seems that many scientists, probably Fredkin himself, consider that "pure" XOR rules are unable to engender anything else but replicators. This is a fact of no doubt, at least for the classical definition for the CA that are structures on regular lattices.

However, from pure physical point of view, there are no serious reasons to state that the homogenous space has to be modeled only by lattice structures; according to us every topologically homogenous cellular structure could be a worthy candidate for a discrete mathematical model of any physical phenomenon. Therefore, our main question in this work is whether there are any homogenous cellular structures, where the "pure" XOR rules possess behavior different from the replicative one. The answer of this question, as we will see later, is positive.

In order to demonstrate our idea we need to re-define some classical notions in the Theory of CA. We introduce the new concept "CA on infinite directed homogenous graph" (unlike the traditional "CA on regular lattice"). We consider the vertices of those graphs as cells, and their oriented edges—as neighboring relations. In this aspect the classical notion of cell neighborhood loses its ordinary meaning, as in this case it is completely possible a cell to be a neighbor to another cell, but the latter not to be a neighbor to the first one. Let us point out too that in this sense every regular lattice can be described as a canonical graph, i.e. the new concept for CA on homogenous digraphs completely comprises the classical concept of CA on regular lattices.

The "trick" we used to acquire the desired attribute of "non-replicativity" is that we consider a very special class of homogenous graphs that we named CP-graphs. The latter possess the extraordinary feature to be composed of topologically *indiscernible* vertices but, in the same time, of topologically *discernible* edges starting from a same vertex. It is interesting to note that even in these conditions the reasons for the appearance of non-replicative behavior remain indistinct and unexpected in a sense. Although there is a set of such non-replicative examples, it is hard to find even one of them.

It might seem that the construction and the analysis of such kind of unusual structures is a fruit only of our academic interest and of our willingness to demonstrate some non-conventional behavior of well studied rules at the expense of non-conventional, but homogenous spaces. As you shall see at the end of our work, however, there is a certain reason behind all this that goes beyond the limits of the demonstration of a mathematical phenomenon for its own sake.

The voluntary refusal from the replicative attribute (i.e. the exact duplication of the primary structure) might be considered illogical, especially after the historical notes for the work of von Neumann. But we consider that the biological process of the evolution is more in a support to this idea. Really, although the process of reproduction of the living organisms always leads to the appearance of same-of-the-kind, it is more a process of resemblance than an act of exact duplication of the prime source. The preservation of some parental attributes and the mutation of others is the basis of the improvement of species and of the appearance of new ones. Furthermore, we would like to admit that in strict rules the exact copying of the parental structure does not exist in nature. The configurations we consider possess exactly this feature of relative resemblance that we think is impressive, especially that we have not been looking for it intentionally.

The things seem far more interesting when related to some quantum-mechanical phenomena. In quantum physics there is the so called CPT-theorem, that says that the physic laws remain unchanged at the simultaneous change of the charge conjugation (C) and the parity (P) of the elementary particles as well as at reversal in time direction (T).

On the other side, the CP-invariance is valid for the electromagnetic and the strong interactions but is violated with the weak interactions (for instance, at the decay of a kaon). Some hypotheses exist, according to which this CP-violation is in the basis of the occurrence of relative asymmetry in nature, starting in micro-world and ending with the strange combination of symmetry and asymmetry in living creatures (the existence of two cerebral hemispheres with different functions, for example).

In this aspect, our rather non-standard interpretation of this phenomenon is a matter of interest. We consider that it is of some sense to examine the CP-violation not as an attribute of the physical laws (the "rules" in our case), but as an attribute of the space (the "cellular structure"). With the existence of examples of "semi-symmetry" of the CP-automata we examine, such an idea does not seem only a notional hypothesis.

All these aspects may seem more or less speculative to the reader, as we do not provide here any serious reason for the fact that we have chosen a specific class of automata and a specific class of rules. There is such a reason but it is too long to be presented here even shortly. We will only point out that we have described it extensively in our work [9].

The aim of the current work, in this sense, is only to introduce our findings of the phenomenon of the relatively chaotic ("non-replicative") behavior of balanced functions in homogenous spaces and its statistic analysis. We will have to introduce new metrics for the latter.

In any case, however, we will consider our goals achieved if this work brings at least some interest to the analysis of the considered by us CA on homogenous directed graphs, despite the motives that drive the potential explorer to them.

2 CA on homogeneous directed graphs—definitions

In this section we will give the formal definitions that we are going to use further on. As the newly introduced here definition for "CA on an infinite homogeneous directed graph", as we have stated above, comprises completely the classical definition for "CA on a regular lattice", in order to make it short, from here on we will always use only "CA", meaning (unless otherwise stated) the more general case.

Definition 1 *(CA)*

The cellular automaton (CA) is called a dynamic discrete structure, built on an infinite homogeneous digraph $\Gamma(V,E)$[1] *in the following way:*

[1]The graph Γ is homogenous when all its vertices V are topologically indiscernible. From physical point of view this means that if we are in such a (uniform) space, then there is no way (without using any coordinate systems, "markers", etc.) that we could determine where exactly (at what position) we are!

1. *We consider all vertices V of the graph Γ as discrete variables that we call cells. We will mark the cells with small Latin letters (usually with c), possibly with low case; the set of all cells will be given with C. The cells may occupy in any specified time unit t one of a finite non-empty set S of values we call states: $S = \{s_0, \dots, s_{k-1}\}$. One of these states, lets say s_0, we consider to be a special state we call blank state. The state of the cell c at the moment t we will describe as $c\,[t] : \; c\,[t] = s_i$, where $s_i \in S$.*

2. *Let we have an introduced fixed order for the cells in C. Then we will indicate the ordered set of the states of all cells in C at the time t, that is called configuration (or generation) with G_t. We call the configuration G_0 initial configuration. We will indicate the set of all possible configurations in CA with G.*

3. *As Γ is homogenous, the number of edges that enter (or exit) every vertex $v \in V$, is the same. Let us mark the number of the incoming edges as n ($indegree(CA) = n$), and the number of the outcoming edges as m ($outdegree(CA) = m$). We will say that the cell $c_1 : \; c_1 \sim v_1$ "depends" on the cell $c_2 : \; c_2 \sim v_2$, where $v_1, \; v_2 \in V$, in case that an oriented edge $e \in E : \; e\,(v_1, v_2)$ exists, starting at v_2 and ending at v_1 (we will also say that c_2 "influences" on c_1). When talking of the CA (instead of the graph Γ) we will call those edges "dependency links" (and correspondingly: "influence links"). We will describe the links with small Latin letters (usually with l), possibly with lower case; we will indicate the set of all links in CA with L. We call neighbors[2] all cells that a certain cell depends on. Thus the number of the neighbors of each cell is exactly n.*

4. *The dynamics of CA is defined by the local function $f \; : \; S^n \to S$, that we call a rule. All cells simultaneously (in parallel) change their state in every discrete moment of time ("tact"), defining their new state corresponding to the state of their neighbors and the transformation defined by the rule, as follows:*

   $$c\,[t+1] \; := f\,(c_1[t], \dots, c_n[t])$$

   where $c_1, \dots, c_n$ are the neighbours of the cell c.

   We call the function $\varphi \; : \; G \to G$, defined by f, a global function for CA.

[2] We introduce this definition on an obvious analogy with the definition of the neighborhood, we know from the traditional "lattice" CA. It is strange enough that the cells "influenced" by a given cell, do not possess a specific name in the classic Theory of CA (because of the simple fact, that with lattices every cell, that is "influenced" by another cell, also always "influences" on the latter).

After this definition let us note that the expansion of the notion "cellular automaton" on the set of infinite homogenous graphs provides us with unexpected possibilities. Really, although the homogenous graphs are uniform structures with topologically indiscernible vertices, it does not mean certainly, that the latter attribute is automatically transferred on their connecting edges.

There is a great number of such graphs with topologically indiscernible as well as with topologically discernible edges, and also ones of the mixed type. We will call the homogenous graphs of the first type (and their corresponding automata) "absolutely homogenous", the ones of the second type—"CP-homogenous" and the ones of the third type—"semi-CP-homogeneous".

On Fig. 3 we have presented the neighbors from Fig. 2 as a homogenous directed graph. According to the newly introduced terminology we can classify the topology in the first case (four neighbors) as absolutely homogenous, while at the second (with five neighbors)—as semi-CP-homogeneous. This simple observation makes us to formulate the hypothesis that, probably, the reason for the more complex dynamics of the CA with five neighbors and the Fredkin's rules lies in the "discernibility" of some of the links that define the neighborhood. We can also ask the question, if this is true, then how the dynamics influences on the topological discernibility of the neighboring links and to what extent the "tip of the balance" goes in favor of the (absolute) CP-homogeneity will increase the manifestations of "chaotic" behavior of the automaton.

Really, in case we consider all forms of symmetry unwanted, we come to the conclusion that the possible asymmetric (and therefore non-replicative) forms of behavior could be provoked only by CA with strictly discernible links. In order to facilitate our presentation from here on we will scrutinize only CP-automata with only two incoming and two outcoming neighboring links for every cell that also meet some additional requirements. As we are not going to scrutinize any other types of CP-automata here, to make it short, we will name "CP-CA" the cellular automata defined in the following paragraph.

Definition 2 *(CP-CA)*
We call a cellular automaton CP-CA, if it meets the following terms:

1. *There are strictly two incoming and two outcoming neighboring links for every cell, i.e.*

   $indegree(CA) = 2, \; outdegree(CA) = 2.$

2. *The links, that go in/out to every cell, are topologically discernible, i.e. it is possible to divide provisionally all links in two groups, that we will name "left" and "right"[3]. We will mark the set of*

[3] Thus in the so defined "CP-space" we do not have at our disposal "special sites" (special points in the space), but we still have

the left-handed links in a given CP-CA as L_L, and the set of the right-handed ones—as L_R, where $L_L \subset L$, $L_R \subset L$: $L_L \cup L_R = L$, $L_L \cap L_R = \emptyset$.

3. *CP-CA is a binary CA—all cells can contain only one bit of information (0 or 1): $k = 2$, $S = \{0, 1\}$.*

4. *The rule of the automaton is* XOR*:*

$$c\,[t+1] \;:= c_1\,[t] \;\; xor \; c_2\,[t]$$

where c_1 and c_2 are neighbors to c, and t is the time.

5. *Let us name the sequence of cells and links $I\,(c_0, l_0, c_1, \ldots, l_{p-1}, c_p)$, where $l_i \in L$: $l_i(c_{i+1}, c_i)$, $i = 0, \ldots, p-1$, an informational channel, and the cells c_0 and c_p—correspondingly a beginning and an end of the channel. Then $\forall c_1, c_2 \in C \Rightarrow \exists I\,(c_1, ..., c_2)$, where c_1—beginning, c_2—end of the channel I[4].*

On Fig. 4–6 we provide examples of cellular automata with CP-topology, that meet the requirements of the definition for CP-CA. In the following section we will analyze their behavior on behalf of their ability for replication.

Due to the specific character of the "parity" rules of our CP-CA, in order to find possible non-replicative examples, it will be sufficient to analyze any finite and non-empty initial configuration. Therefore, in all provided examples, at the beginning of the process ($t = 0$) all cells are 0 (zero), but one that is 1 (one), or, in other words, we have chosen for CP-CA a fixed initial configuration, that comprises only one nonblank cell.

3 Analysis of CP-CA

The analysis of the behavior of the automata at Fig.4–6 would be difficult unless we have a geometrical way to illustrate the results we come to. Such problems do not exist with the "classical" CA on regular lattices, because here we usually deal with one or two-dimensional (mostly square, some times hexagonal or triangular) lattices. The visualization of CA on infinite homogenous digraphs, however, (that can be of a rather bizarre structure, as seen from the examples), can raise considerable difficulties, especially if we want to illustrate their dynamics for generations of a higher number.

Therefore, the examples on Fig. 4, 5 are not accidental but well chosen. It turned out that with some imagination we can illustrate the CA on Fig. 4 (we call it "CP-MOSAIC") on an infinite square lattice, while preserving the distance between neighboring cell in the automaton almost the same. Such presentation of CP-MOSAIC as a "lattice with holes" is given on Fig. 7a.

We can put down this presentation strictly symbolically in the following way:

$$c_{x,y}\,[t+1] := \begin{cases} c_{x-1,y-1}\,[t] \oplus c_{x,y+1}\,[t]\,, & \text{if (3)} \\ c_{x,y-1}\,[t] \oplus c_{x+1,y+1}\,[t]\,, & \text{if (4)} \\ c_{x-1,y+1}\,[t] \oplus c_{x+1,y}\,[t]\,, & \text{if (5)} \\ c_{x-1,y}\,[t] \oplus c_{x+1,y-1}\,[t]\,, & \text{if (6)} \\ \text{not defined,} & \text{otherwise} \end{cases} \tag{2}$$

where $c_{i,j}\,[t]$ is the value at moment t of the square cell from Fig. 7a with coordinates (i, j), $i, j \in \mathbf{Z}$ and the conditions are as follows:

$$x \equiv 0(mod\ 3),\;\; y \equiv 1(mod\ 3) \tag{3}$$

$$x \equiv 0(mod\ 3),\;\; y \equiv 2(mod\ 3) \tag{4}$$

$$x \equiv 1(mod\ 3),\;\; y \equiv 0(mod\ 3) \tag{5}$$

$$x \equiv 2(mod\ 3),\;\; y \equiv 0(mod\ 3) \tag{6}$$

There is also such presentation for the class of automata from Fig. 5, that we call "CP-CYLINDER". In this case it is convenient to present every element of this class—CP-CYLINDER (p), where $p = 2k$, $\;k \in \mathbf{Z}^+$—as p number of rows in a square lattice on a plane (see Fig. 7b). In this case the corresponding symbolic presentation is:

$$c_{x,y}\,[t+1] := \begin{cases} c_{x-1,0}\,[t] \oplus c_{x,p-1}\,[t]\;, & \text{if (8)} \\ c_{x-1,y}\,[t] \oplus c_{x,y-1}\,[t]\;, & \text{if (9)} \\ c_{x+1,y}\,[t] \oplus c_{x,y-1}\,[t]\;, & \text{if (10)} \end{cases} \tag{7}$$

where $c_{i,j}\,[t]$ is the value at time t of the square cell from Fig. 7b with coordinates (i, j), $i = 0, \ldots, p-1$, $j \in \mathbf{Z}$ and following conditions:

$$y = 0 \tag{8}$$

$$y \equiv 0(mod\ 2),\;\; y > 0 \tag{9}$$

$$y \equiv 1(mod\ 2) \tag{10}$$

the notion for "direction" ("left" and "right"). Naturally, it is relative which links we will call "left" and which—"right".

[4] The idea of this restriction is, that every cell of the automaton can (in principle) exchange information with any other cell of the same automaton.

In this case we can consider (8) as kind of a boundary condition for this type of CA. Let us also note, that due to their linear nature these structures can be described even as one-dimensional automata with 2^p states and radius 1 (i.e. as $(2^p,\ 1)$–automata in the Wolfram's notation—see, for example, [4]).

Let us now see what the behavior of CP-CA on Fig. 7 is. We can hope that the topological discernability of the links of those otherwise homogenous automata will be sufficient misbalance that leads to some more interesting, non-replicative behavior.

But those expectations proved to be excessive. Even clever examples as the CP-MOSAIC from Fig. 7a, despite the CP-topology of their structure, are incapable to bring to life anything else but replicators. The dynamics of the first several generations of CP-MOSAIC with initial configuration of a lonely non-empty cell is given on Fig. 8. It is obvious that after the initial few tacts four groups are formed, each consisting of two closely situated non-empty cells, that are "drifting away" from the imaginary center, as in the rather less complicated Fredkin's example on Fig. 2a on a regular lattice. The only thing that differs to some extent our "new" example from the "classical" one, is the unusual period of replication as well as the weird replicative "pattern" comprising two but not one cell.

The analysis of the first automaton from the CP-CYLINDER class, namely the CP-CYLINDER (2), proved to be even more disappointing (see Fig. 9a). What we get at last is practically identical to the well known Triangle of Sierpinski. The latter might also be presented as one-dimensional (2, 1/2) automaton of Wolfram or retrieved from the Triangle of Pascal modulo two.

The example of the next CP-CA from the examined class of automata is identical. CP-CYLINDER (4) is a trivial replicator again, similar to the Triangle of Sierpinski, whose dynamic development in time is illustrated on Fig. 9b. The only thing that impresses to some extent is that some generations of this automaton show an inclination to asymmetry, which, however, is quickly "dissolved" by the approaching replicative figures, to establish perfect symmetry again.

At this point, the already bored reader is probably grumbling, that we occupy him with some very trivial examples of linear automata. Really, the one-dimensional automata are well studied, especially for small values of the radius and the states of the cells in them, and the "parity" rules are well known as stubborn originators of replicators. This makes it easy to come to the hypothesis that all CA of the so defined by us CP-CYLINDER class will be replicators that may be very similar to the Triangle of Sierpinski.

But just have a look at CP-CYLINDER (6): it is even hard to believe, but what we get here *is not* a replicator (see Fig.10)! What's more—it is obvious that all configurations (but the initial one, of course) show a vague but remarkable tendency to asymmetry. This result seems even more striking after the just analyzed trivial examples of a lower series in CP-CYLINDER and when we recognize that it is a matter of one-dimensional automata.

The further analysis of the CP-CYLINDER class brings the following: the next CP-CYLINDER (8) in the series is a replicator, then comes a series of three non-replicative automata, CP-CYLINDER (16) is a replicator again, and so on. Generally speaking, it turns out that CP-CYLINDER (p) for $p = 2^k,\ \ k \in \mathbf{Z}^+$ is a replicator, and vice versa, when p is not a power of two, then the corresponding CA of that class has non-replicative behavior.

Until now we tacitly used some intuitive "geometrical" notion for "replication" that helped us to analyze CP-MOSAIC and CP-CYLINDER. This "notion" could be very useful in case we possess "visual" presentations of the CA we have examined as the ones on Fig. 7, for example, but, unfortunately, this is not always possible. For highly branched structures as the ones of the CP-L-R class (see Fig. 6) obviously such presentation does not exist. Furthermore: as structures of such kind tend to grow extremely with the number of their generations (see below), any "visual" presentations of their dynamics, inclusive the one that would ignore the idea of the "relative retaining of neighborhood" becomes simply impossible. Therefore, from now on we will try to provide a sufficient number of sensible and effective statistical criteria, that could provide a safe way for the analysis of all CP-CA not using visualization of the structures obtained.

The process of defining the notion "replicative pattern" that is an innate attribute of the process of replication, with the so given definition of a CA, proves to be a rather complex task to be worth examining it here in detail. Therefore, instead of trying to define a notion for uniformity of a "pattern", translated to different positions in the discrete space, defined by an infinite homogenous digraph, we will ignore its form completely and will be satisfied only with the number of the non-blank cells in it.

Definition 3 *(Population)*
The number of non-blank cells in a certain generation of a given cellular automaton CA is called a population:

$$Pop(G_t) = \sum_i c_i\,[t]\ ,\ \ c_i\,[t] \neq 0$$

where G_t is a generation at the moment t of the cellular automaton CA.

Let us note now that despite of the period and the form of our "replicative pattern", if a given CA has a replicative behavior, then after some time interval, from any moment on, a situation arises where the population is a fixed number. We can define the following criterion:

Criterion 1. *Let CA is a cellular automaton with replicative behavior. Then*

$$\exists M \in \mathbf{Z}^{+} : \forall t \in \mathbf{N}\ \exists t' \in \mathbf{N},\ t' \geq t \Rightarrow Pop\,(G_{t'}) = M$$

where t, t' are discrete time cycles.

We have provided the history of populations of a typical replicator on Fig. 11a. We can make an instant comparison with the populations of the first two non-replicative automata of the CP-CYLINDER (Fig. 11b,c) and CP-L-R (Fig. 11d) classes correspondingly. It is obvious that with CA of non-replicative behavior the function defining the population, although not monotonous as general, tends to relative growth.

The criterion we chose for the previous paragraphs, however, could raise some suspicions, especially for cases with highly branched topologies as the ones of CP-L-R, for example. It might be expected that with the automata of the latter class there will be ones with replicative behavior, that will be increasing the number of replicative figures too fast, although this would not lead to the "pulsations" of the population we encountered with the linear automaton on Fig. 11a.

At the analysis of the first detected by us CA with non-replicative behavior (namely CP-CYLINDER (6)) we mentioned that this automaton shows an obvious tendency towards a weak but notable asymmetry. In order to actuate the mechanism of replication it is necessary to reach a situation from a certain moment on when there is a double or multiple copying of the same replicative "pattern" in different positions of the discrete space of a given CA. What remains unclear is the answer to the question how to distinguish and identify those "patterns", and whether they are a stable or only an ephemeral manifestation of symmetry.

By its own the definition of geometric symmetry as a function of space for CP-structures as general seems to be a complex task. Therefore, instead of using some arguable and obscure notion of "geometric symmetry", we will define the notion "symmetry" for cellular automata as a function of the dynamics of a given CA.

Definition 4 *(Meaningful cells)*

Let $c \in C$ is a cell of a given cellular automaton CA with fixed initial configuration G_0. We will say that c is meaningful for time t with fixed G_0, if:

$$\exists t' : t' \leq t,\ t, t' \in \mathbf{N} :\ c[t'] \neq 0$$

where $t,\ t'$ are time cycles, and 0 (zero) is the blank state of cells in CA. We will mark the set of all meaningful cells for time t with fixed G_0 as $\dot{C}_t(G_0)$.

Definition 5 *((Dynamically) symmetrical cells)*

Let $c_1, c_2, \ldots, c_n \in \dot{C}_t(G_0)$ make n ($n \geq 2$) in number different meaningful cells of a given CA. We will say that those cells are (dynamically) symmetrical for time t with fixed G_0, if:

$$\forall t' : t' \leq t,\ t, t' \in \mathbf{N} \Rightarrow c_1[t'] = c_2[t'] = \ldots = c_n[t']$$

where $t,\ t'$ are time cycles. We will name the set of all cells in CA, that possess own symmetrical analogues for time t with fixed G_0, "set of the symmetric cells in CA for time t". We will mark this set as $\ddot{C}_t(G_0)$:

$\ddot{C}_t(G_0) = \{c_1 \in \dot{C}_t(G_0) : \forall t' \leq t,\ t, t' \in \mathbf{N},\ \exists c_2 \in C,$
$c_1 \neq c_2 : c_1[t'] = c_2[t'],\ c_i[0] \in G_0\}.$

Definition 6 *((Dynamic) symmetry)*

We will call the correlation of all symmetric cells towards the meaningful ones for t at fixed G_0 of a given CA (dynamic) symmetry and will mark it as $Sym(G_0,\ t)$:

$$Sym(G_0,\ t) = \frac{card\left[\ddot{C}_t(G_0)\right]}{card\left[\dot{C}_t(G_0)\right]}$$

where $card\left[\ddot{C}_t(G_0)\right]$ is the number of all symmetrical, and $card\left[\dot{C}_t(G_0)\right]$ —the number of all meaningful cells for time t with initial configuration $G_0 \in CA$.

Let us note now that for a CA with replicative behavior, the number of meaningful cells without symmetric analogues must be restricted from above. Here we come to the next criterion:

Criterion 2. *Let CA has a replicative behavior. Then:*

$$\exists M \in \mathbf{N} :\ \forall t \in \mathbf{N} \Rightarrow\ card\left[\dot{C}_t(G_0)\right] - card\left[\ddot{C}_t(G_0)\right] \leq M$$

where t is the time, and G_0 is a fixed initial configuration in CA.

On the other hand, the permanent presence of asymmetric cells is a certain sign of the appearance of non-replicative behavior, that leads us to the following simple corollary:

Corollary 1 *In case for the fixed $G_0 \in CA$*

$$Sym(G_0, t) \overset{t\to\infty}{\longrightarrow} r < 1$$

where t is the time, and $r \in \mathbf{R}$, then CA possesses non-replicative behavior.

We have illustrated this corollary on Fig.12 with the symmetry function of the first two non-replicative automata of the CP-CYLINDER class. Note that the dynamic symmetry of those two automata in about 1000 generations falls closely to 50% and afterwards remains stable at this level.

It would be interesting to investigate how really this "parity" between symmetry and asymmetry of those automata is achieved. We have done this on Fig. 13, where we have created a "symmetry map" for CP-CYLINDER (6) for $t = 50$ and $G_0 : Pop(G_0) = 1$. We have presented the symmetric cells of CP-CYLINDER with solid black color, the asymmetric ones—in gray, and the cells, which have never been 1 in the first 50 cycles (i.e. the ones that are not meaningful), are left blank. This picture does not change significantly with the time and remains almost identical also for CP-CYLINDER (10).

The simple analysis shows that the cells of the odd rows of those two automata tend to asymmetry, while the ones of the even rows (but the column with the only non-blank cell in the initial configuration G_0) have their symmetric analogues. In general, a more profound analysis would show that for CP-CYLINDER (6) and CP-CYLINDER (10) the following relation exists for $t \to \infty$:

$$\begin{array}{l} c_{x,y} \in \ddot{C}_t(G_0) : c_{x,y}[t] = c_{-x,y}[t], \forall t \in \mathbf{N}, \text{ if } (12) \\ c_{x,y} \notin \ddot{C}_t(G_0), \text{ if } (13) \end{array} \tag{11}$$

$$x \neq 0,\ y \equiv 0\ (mod 2) \tag{12}$$

$$x = 0 \text{ or } y \equiv 1\ (mod 2) \tag{13}$$

However, as it usually happens with complex dynamic structures as the CA, all general conclusions are usually wrong. Not going in deep detail, we will only note that the correlation (11) is not applicable for all automata of that class, as we might be tempted to suggest. For the very next non-replicative automaton, namely CP-CYLINDER (12), the attempt for a conclusion "on analogy" fails: the general symmetry here is not even 50% any more...

We will finish this section with a short analysis of the CP-L-R class automata. With them the **Corollary** of **Criterion 2** is not applicable as the fast growth of the population leaves far behind the growth of the asymmetry. Therefore, to prove that those automata are non-replicative, we have illustrated on Fig.14 the growth of the (absolute) number of asymmetric cells, and not the fall of the (general) symmetry as we have done for the CP-CYLINDER class. The continuous, although slow, emerging of asymmetric cells in CP-L-R is a certain proof that those automata ultimately have a non-replicative behavior.

4 Attributes of the CA with Fredkin's rules

In the following section we will prove several theorems for the attributes of the CP-CA we have analyzed. Some of these results are easily generalized for CP-automata (of the more general type) or for CA with Fredkin's rules as a whole, but we prefer their more particular formulation for reasons of better clarity.

The first two theorems are just curious or elegant but the last (third in the row) under other circumstances could have been obvious. However, the final result is of real interest—it could have been trivial unless the non-replicative examples for CP-CA. This remarkable possibility that we study in this work is the one that makes things considerably different. The combination of the "trivial" result of Theorem 3 and the "non-trivial" examples of non-replicative CP-CA makes an explosive mixture that can bring us to far leading general conclusions of metaphysical nature that are shown in [9].

All CP-CA attributes we are going to prove here are limited within the subset of the finite configurations. As we are not going to look into other configurations but the finite ones, we are not going to mention this onwards.

Definition 7 *(Finite configuration)*
We will say, that $G' \in G$ is a finite configuration, if $Pop\,(G') < \infty$. We will describe the set of all finite configurations as $\bar{G}$.

Definition 8 *(Garden-of-Eden)*
Let $G' \in G$ is a configuration of a certain CA. We will say, that G' is a Garden-of-Eden for CA, if $\nexists G'' \in G :\ G'' \xrightarrow{\varphi} G'$, where φ is the global function of the automaton. We will describe the set of all configurations that are Gardens-of-Eden for CA as $GOE(G)$.

In case of no existence of a finite configuration that brings $G' \in \bar{G}$, we will say that G' is a Garden-of-Eden for finite configurations. The set consisting of all finite Garden-of-Eden configurations is described as $GOE(\bar{G})$:

$$GOE(\bar{G}) = \left\{ G' \in \bar{G} : \nexists G'' \in \bar{G},\ G'' \xrightarrow{\varphi} G' \right\}$$

Definition 9 *(Reversible cellular automaton—RCA)*
We say that CA is a reversible[5] cellular automaton (RCA) in case that every CA configuration has no more than one predecessor:

$$\forall G' \in G \Rightarrow \left\{ \begin{array}{l} \nexists G'' \in G : G'' \xrightarrow{\varphi} G' \\ \nexists G'', G''' \in G,\ G'' \neq G''' : G'', G''' \xrightarrow{\varphi} G' \end{array} \right.$$

i.e.:

$$\forall G' \in G \Rightarrow \left\{ \begin{array}{l} G' \in GOE(G) \\ \exists! G'' \in G : G'' \xrightarrow{\varphi} G' \end{array} \right.$$

[5]In some works one can find the term "injective".

We say that CA is reversible for finite configurations if:

$$\forall G' \in \bar{G} \Rightarrow \begin{cases} G' \in GOE(\bar{G}) \\ \exists! G'' \in \bar{G} : G'' \xrightarrow{\varphi} G' \end{cases}$$

We will describe the set of all reversible CA as RCA, and the set of all reversible CA for finite configurations as $\overline{RCA}$.

Definition 10 *(Invertible cellular automaton—ICA)*
We say that CA is a invertible[6] *cellular automaton (ICA) in case that every CA configuration has only one predecessor:*

$$\forall G' \in G \Rightarrow \exists! G'' \in G : \ G'' \xrightarrow{\varphi} G'$$

We say that CA is invertible for finite configurations if:

$$\forall G' \in \bar{G} \Rightarrow \exists! G'' \in \bar{G} : \ G'' \xrightarrow{\varphi} G'$$

By analogy we will describe the set of all invertible CA as ICA, and the set of all invertible CA for finite configurations as $\overline{ICA}$.

Let us now investigate CP-CA for reversibility regarding the subset of the finite configurations. The examples on Fig. 1 and 2 brought us the idea that probably all CA with Fredkin's rules (or, at least those of them with replicative behavior) are reversible. Actually, every initial configuration possesses the ability to copy itself infinitely in the space, thus giving us the notion that the whole process is reversible.

This notion is very obsessive one, but it is wrong. Moreover, it was proved that for CA (on homogenous digraphs) neither the process of replication, nor its absence have anything in common with the "reversibility". This is depicted on Fig.15 where we have shown that *all* CA from the CP-CYLINDER class *are not* reversible. Using (7) we can express the above statement symbolically as:

$G'_0 = \{c_{x,y}[0] = 0,\ x \in \mathbf{N},\ y = 0, \dots, p\ /\ c_{0,0}[0] = 1\}$
$G''_0 = \{c_{x,y}[0] = 0,\ x \in \mathbf{N},\ y = 0, \dots, p\ /\ c_{i,1}[0] = 1,$ $i = 1, 3, \dots, p-1,\ c_{j,1}[0] = 1, j = 2, 4, \dots, p-2\}$
$G_1 = \{c_{x,y}[1] = 0,\ x \in \mathbf{N},\ y \in 0, \dots, p\ /\ c_{1,0}[1] = 1,$ $c_{0,1}[1] = 1\}$
$\forall p = 2k,\ k \in \mathbf{Z}^+,\ G'_0 \xrightarrow{\varphi} G_1, G''_0 \xrightarrow{\varphi} G_1 \Rightarrow$
$\Rightarrow$ CP-CYLINER$(p) \notin \overline{RCA}$.

In the previous paragraph we made an extensive statistical examination of the CA of the CP-CYLINDER class and proved that among them one can find ones with replicative behavior as well as others with non-replicative one. With this counterexample we essentially prove that non-reversible CP-CA exist as well as that the reversibility of the latter is by no means related to the replication.

It would be useful to be of disposal of a simple criterion that lets us to find out whether a certain CP-CA is reversible or not, without looking for specific examples. Further on we will describe such a criterion; for that let us remind some classical concepts from the Theory of Graphs and to define some their analogues for CA as well as some new definitions.

Definition 11 *(Chain; simple chain; cyclic chain (cycle))*
Let $\Gamma(V, E)$ is a directed graph. The finite or infinite sequence of vertices and edges:

$$W(\dots, v_0, e_1, v_1, \dots, v_{p-1}, e_p, v_p, \dots),\ v_i \in V,\ e_i \in E$$

will be named a chain, if $e_i : e_i(v_{i-1}, v_i)$ or $e_i : e_i(v_i, v_{i-1})$ and every edge $e_i \in W$ joins W only once.

We say that W is a non-crossing itself (or simple) chain in case that every vertex of Γ is incidental of no more than two edges of W.

Let W is a finite chain: $W(v_0, e_1, v_1, \dots, v_{p-1}, e_p, v_p)$. We say that W is a cyclic chain (or a cycle) in case of $v_0 \equiv v_p$.

When talking of CA, built on the graph , we will use the same definitions only changing the corresponding vertices with cells and edges with links, i.e.:

$$W(\dots, c_0, l_1, c_1, \dots, c_{p-1}, l_p, c_p, \dots),\quad c_i \in C,\ l_i \in L$$

is a chain for CA.

Definition 12 *(Reversible chain for CP-CA)*
Let $c_{1,0} \in C$ is a cell from a given CA ∈CP-CA. The chain W:

$$W(\dots, c_{0,-1}, l^L_0, c_{1,0}, l^R_0, c_{0,0}, \dots, c_{0,j-1}, l^L_j, c_{1,j}, l^R_j, c_{0,j}, \dots)$$

where $c_{i,j} \in C$, $i = \{0, 1\}$, $j \in \mathbf{Z}$, $l^L_j \in L_L$: $l^L_j(c_{1,j}, c_{0,j-1}), l^R_j \in L_R : l^R_j(c_{1,j}, c_{0,j})$ is called a reversible chain for CP-CA, described by $c_{1,0}$.

We will name the cells $c_{0,j} \in W$, $j \in \mathbf{Z}$ "row of influence" for the reversible chain W, and $c_{1,j} \in W$, $j \in \mathbf{Z}$— "row of dependence" for W.

In case $c_{1,0} \equiv c_{1,p}$ for some $p \in \mathbf{Z}$, $p \neq 0$, W is a cyclic reversible chain. If $c_{0,p} \equiv c_{1,q}$ for some $p, q \in \mathbf{Z}$, W is a self-crossing chain.

We have provided examples of different reversible chains on Fig. 16. Actually, there are no self-crossing reversible chains in CP-CA, that is going to make easier the proof of some statements hereafter.

Lemma 1 *(No self-crossing reversible chains exist in CP-CA)*
Let CA ∈ CP-CA and $W(..., c_{0,-1}, l^L_0, c_{1,0}, l^R_0, c_{0,0}, ..., c_{0,j-1}, l^L_j, c_{1,j}, l^R_j, c_{0,j}, ...)$ is a reversible chain from CA. Then $\not\exists c_{0,p}, c_{1,q} \in W$: $c_{0,p} \equiv c_{1,q}$, where $p, q \in \mathbf{Z}$.

[6] Called also "bijective".

Proof. Let us suggest, that $\exists c_{0,p}, c_{1,q} \in W : c_{0,p} \equiv c_{1,q}$ for fixed $p, q \in \mathbf{Z}$. We will prove that then:

$$c_{0,p+j} \equiv c_{1,q+j}, \forall j \in Z \tag{14}$$

Really, as CP-CA are built on homogenous graphs (according to the definition of the CA), then every topological attribute of a certain vertex of a given homogenous graph is an attribute of all the vertices of the same graph. What is more, as those graphs for CP-CA possess topologically discernible edges (CP-graphs), we can introduce an original coordinate system that can help us to determine the position of any cell of the automaton.

For this purpose we fix a cell as a beginning of the coordinate system and then build an arbitrary chain from the "beginning" to a random cell of a given CP-CA. Then the orientation and the affiliations of the links to any of the two classes ("left" or "right") define precisely the position of the last cell towards the first (fixed) cell; this very technique is defined and used further on.

Let $c_0 \in C$ is a fixed cell, $CA \in$ CP-CA and let W_1 is a random cycle (not exactly reversible): $W_1(c_0, l_0, c_1, ..., c_p), \quad c_0 \equiv c_p$.

Let us now fix another cell $c'_0 \in C$ of CA and let $W'_1(c'_0, l'_0, c'_1, \ldots, c'_p)$ is a chain, defined in the following way:

$$l'_i : \begin{cases} l'_i(c'_{i-1}, c'_i), & \text{if} \quad l_i : l_i(c_{i-1}, c_i) \\ l'_i(c'_i, c'_{i-1}), & \text{if} \quad l_i : l_i(c_i, c_{i-1}) \\ l'_i \in L_L, & \text{if} \quad l_i \in L_L \\ l'_i \in L_R, & \text{if} \quad l_i \in L_R \end{cases} \tag{15}$$

Then W'_1 is a cycle too: $c'_0 \equiv c'_p$, because otherwise c_0 and c'_0 would have been topologically discernible.

The statement (14) is an exception of $W_1 \subset W$: $W_1(c_{0,p}, l^L_{p+1}, c_{1,p+1}, l^R_{p+2}, ..., c_{1,q})$ and $W'_1 \subset W$: $W'_1(c_{0,p+i}, l^L_{p+i+1}, c_{1,p+i+1}, l^R_{p+i+2}, ..., c_{1,q+i})$.

But in case of (14), every cell of the W chain would have had two incoming and outcoming links and, therefore, there are no more CA cells out of the reversible chain W (see clause 5 of the definition for CP-CA). Thus $CA \in$ CP-CA consists of the only reversible chain that is not cyclic, because the CA are structures on infinite graphs.

An example of such a CA, comprising an only chain W of the kind described in (14), is given on Fig. 17. However, the CA of this kind do not meet clause 5 of the definition for CP-CA.

Here we come to a discrepancy with the suggestion we made at the beginning of the evidence that brings us to the statement in the Lemma. ■

Theorem 1 *(The necessary and sufficient condition for the reversibility of CP-CA)*

Let W is a reversible chain for $CA \in$ CP-CA. Then:

$$W_i \textit{ is not cyclic} \Leftrightarrow CA \in \overline{RCA}.$$

Proof. For the proof of the theorem we will use the following simple statements:

1. According to the Lemma, all reversible chains in CP-CA are simple, therefore we can limit our investigations to the cases, described in Fig. 16 a, b.
2. All reversible chains in CP-CA are identical (isomorphous) because of (15), therefore it is enough to examine any of them.
3. Every cell of a given CP-CA takes part in a single row of dependence and single row of influence. Thus we can describe every configuration as a set of non-intersecting rows of dependence (or rows of influence).
4. The states of the cells of the row of dependence for a given reversible chain at the moment t are defined only by the states of the cells of the row of influence of the same reversible chain at the moment $(t-1)$ and is independent of other cells of the automaton.

Further down we will prove that, in case where W is a reversible chain for CA, then:

$$W_i \text{ is cyclic} \Rightarrow CA \notin \overline{RCA} \tag{16}$$

$$W_i \text{ is not cyclic} \Rightarrow CA \in \overline{RCA} \tag{17}$$

Let us first prove (16). It will be sufficient to show, for example, that the empty configuration $G_t \in \overline{G} : Pop(G_t) = 0$ possess other finite configurations–predecessors but its own self.

Indeed, let $W_i(c^i_{0,0}, l^L_1, c^i_{1,1}, l^R_2, ..., c^i_{0,n})$, $c^i_{0,0} \equiv c^i_{0,n}$ is a cyclic reversible chain of CA. We can define the configuration $G_{t-1} : G_{t-1} = \{c^k_{0,j}[t-1] = 0 \in W_k,\ k \neq i\ /\ c^i_{0,j}[t-1] = 1 \in W_i\}$, for which the cells at the moment $(t-1)$ are 0 (blank), but the cells from the row of influence for W_i , which are set to 1. Then the cells from the row of dependence of every reversible chain at the moment t will be 0 too:

$c^i_{0,j}[t-1] = 1,\ \forall c^i_{0,j} \in W_i \Rightarrow c^i_{1,j}[t] := c^i_{0,j-1}[t-1] \oplus c^i_{0,j}[t-1] = 1 \oplus 1 = 0,\ \forall c^i_{1,j} \in W_i$;

$c^k_{0,j}[t-1] = 0,\ \forall c^k_{0,j} \in W_k,\ k \neq i \Rightarrow c^k_{1,j}[t] := c^k_{0,j-1}[t-1] \oplus c^k_{0,j}[t-1] = 0 \oplus 0 = 0,\ \forall c^k_{1,j} \in W_k$.

As W_i is cyclic, then it possesses a finite number of cells and therefore G_{t-1} is also a finite configuration, which is projected in the empty configuration.

Let us prove (17). If W_i is not cyclic, then W_i comprises an infinite number of cells:
$W_i(..., c^i_{0,-1}, l^L_0, c^i_{1,0}, l^R_0, c^i_{0,0}, ..., c^i_{0,j-1}, l^L_j, c^i_{1,j}, l^R_j, c^i_{0,j}, ...)$.

We will demonstrate that if $G_t \in \overline{G}$ is a finite configuration in CA, then it is either a Garden-of-Eden, or has only one predecessor.

It is not difficult to notice that the cell state at the row of dependence for W_i at the moment t defines only two possibilities for the cell states at the row of influence of the same reversible chain at the moment $(t-1)$.

If, in order to make it short, we introduce $R_i(j) = \sum_{k<j} c^i_{1,k}[t]$, then for the row of influence of the chain W_i at the moment $(t-1)$ we have the following possibilities:

$$c^i_{0,j}[t-1] = \begin{cases} 0, & \text{if } R_i(j) \equiv 0 \ (mod\ 2) \\ 1, & \text{if } R_i(j) \equiv 1 \ (mod\ 2) \end{cases} \tag{18}$$

or the contrary:

$$c^i_{0,j}[t-1] = \begin{cases} 1, & \text{if } R_i(j) \equiv 0 \ (mod\ 2) \\ 0, & \text{if } R_i(j) \equiv 1 \ (mod\ 2) \end{cases} \tag{19}$$

We can see now that of those two possibilities, only (18) could possess a finite number of ones in the so defined row of influence. Really, for (19) we have an infinite number of ones on the left:

$$c^i_{0,j}[t-1] = 1, \ \forall j \in \mathbf{Z}, \ j < p$$

where p is the number of the first nonblank cell of the row of dependence:

$$p = \begin{cases} \min\left\{p : c^i_{1,p}[t] = 1\right\}, & \text{if } \exists c^i_{1,p}[t] = 1 \\ \infty, & \text{if } \nexists c^i_{1,p}[t] = 1 \end{cases}$$

Thus, using (18) for every reversible chain of a given CA, we can define synonymously a predecessor $G_{t-1} \in G$ of $G_t \in \overline{G}$. Besides, if $G_{t-1} \in \overline{G}$, then G_{t-1} is the only finite predecessor for G_t, in case that such one exists.

The statement of the theorem evolves from the proofs of (16) and (17). ■

Taking into account the first part of the above proof—(16), let us go back for a moment to our example in Fig. 15. The logic of this concept is already clear: the non-blank cells of G'_0 and G''_0 occupy the same row of influence, while only the cell state $c_{0,0}[t]$ is inverted from 0 to 1, unlike (16).

It is not a problem to check which of the CP-CA analyzed in §3 possess cyclic reversible chain. Except CP-CYLINDER, it was found that CP-MOSAIC (see Fig. 4) possesses such cycles. The CP-L-R class from Fig. 6, however, is an exception, that immediately leads us to the conclusion that all automata of this type are reversible.

Theorem 2 *(The necessary and sufficient condition for the Garden-of-Eden in CP-CA)*

Let $G_t \in \bar{G}$ is a finite configuration in $CA \in$ CP-CA. Then a given configuration is a Garden-of-Eden if and only if the row of dependence of some reversible chain possesses an odd number of nonblank cells:

$G_t \in GOE(\bar{G}) \Leftrightarrow \exists W_i$—a reversible chain in CA: $\sum_j c^i_{1,j}[t] = 2k+1, \ k \in \mathbf{N}$

Proof. *Necessity.* Let us assume, that:

$\exists G_t \in GOE(\bar{G}) \Rightarrow \forall W_i$—reversible chain in CA: $\sum_j c^i_{1,j}[t] = 2k, \ k \in \mathbf{N}$

i.e. G_t is a Garden-of-Eden and all reversible chains in CA have an even number of nonblank cells in their row of dependence for G_t.

Let us define, however, the row of influence for W_i according (18), despite the fact whether W_i is cyclic or not.

Thus we simply define the cell state at the row of influence at the moment $(t-1)$, while the number of the non-blank cells of this row is finite. Really, if $\exists c^i_{1,j} : c^i_{1,j}[t] = 1$ and we define:

$$p = \min\left\{p : c^i_{1,p}[t] = 1\right\}, \ q = \max\left\{q : c^i_{1,q}[t] = 1\right\} \tag{20}$$

then $c^i_{0,j}[t-1] = 0, \ j < p$ (as $R_i(j) = 0$) and $c^i_{0,j}[t-1] = 0, \ j \geq q$ (as $R_i(j) = 2k \equiv 0 \ (mod 2)$).

On the contrary: if $\nexists c^i_{1,j} : c^i_{1,j}[t] = 1 \Rightarrow c^i_{1,j}[t] = c^i_{0,j}[t] = 0$, or the number of the nonblank cells from the row of influence is finite again (zero).

Defining in this way the state of the cells from the row of influence of every reversible chain W at the moment $(t-1)$, we actually define the configuration $G_{t-1} \in \overline{G}$: $G_{t-1} \xrightarrow{\varphi} G_t$. But this is in discrepancy with the initial suggestion that $G_t \in GOE(\overline{G})$, from which the necessity follows.

Sufficiency. We will demonstrate, that:

$\exists W_i$—a reversible chain in CA: $\sum_j c^i_{1,j}[t] = 2k+1, \ k \in \mathbf{N} \Rightarrow G_t \in GOE(\bar{G})$.

Then we can always define p and q as in (20), because the row of dependence comprises at least one nonblank cell. We will look at the two possible cases for W_i:

1. W_i is not cyclic. Then for the row of dependence we get the only possibility (18), because otherwise G_t will be an infinite, not a finite configuration. In this case:

   $c^i_{0,j}[t-1] = 1, \ j \geq q$ (because of $R_i(j) = 2k+1 \equiv 1 \ (mod\ 2)$),

   from which follows, that G_t can have only infinite configurations for predecessors $\Rightarrow G_t \in GOE(\bar{G})$.

2. W_i is cyclic: $W_i(c^i_{0,0}, l^L_1, c^i_{1,1}, l^R_2, ..., c^i_{0,n})$, $c^i_{0,0} \equiv c^i_{0,n}$. Then we already can specify the row of influence for W_i as in (18), so as in (19). Despite of which possibility we will choose, we reach the following contradiction:

   $c^i_{0,0}[t-1] \neq c^i_{0,n}[t-1]$ (as $R_i(0) = 0$, $R_i(n) = 2k+1 \equiv 1\ (mod\ 2)$),

   but $c^i_{0,0} \equiv c^i_{0,n}$. Therefore in this case even $G_t \in GOE(G)$, as G_t has neither finite, nor infinite configurations for predecessors. ■

Corollary 2 *(Sufficient condition for the Garden-of-Eden in CP-CA)*
Let $G_t \in \bar{G}$ is a configuration in $CA \in$ CP-CA. Then, if:

$$Pop(G_t) = 2k + 1,\ k \in \mathbf{N} \Rightarrow G_t \in GOE(\bar{G})$$

i.e. all configurations in CP-CA, that have odd population, are Gardens-of-Eden for that type of CA.

Proof. If $G_t \in \bar{G}$ possesses odd population, then G_t comprises an odd number of rows of dependence with an odd number of nonblank cells. According to the Theorem, the presence of at least one such row of dependence is sufficient, so as to be able to state that $G_t \in GOE(\bar{G})$. ■

Corollary 3 *(CP-CA are not invertible)*
$\forall CA \in$ *CP-CA* $\Rightarrow CA \notin \overline{ICA}$.

Proof. The statement is obvious: all cellular automata, that meet the definition of CP-CA, comprise Gardens-of-Eden (according to the previous corollary, these are, for example, all configurations with odd population), therefore this kind of automata can not be invertible. ■

Corollary 1 is good to be seen. It was proved that all CP-CA possess a GOE, that consists of a single nonblank cell! Bearing in mind the final non-triviality of the problem with the GOE for CA on regular lattices (see, for example, [10]-[13]), the presence of a minimal configuration with such characteristics for automata with such rather complex dynamics, might be a find[7].

Corollary 2 is quite obvious as to be of any mathematical interest. The invertible CA, however, have a long (almost a quarter of a century) history of falls and raises and nowadays they are one of the "hot spots" for research in the modern mathematical physics [14], [15]. A topic of special interest is the presence of reversible, but not invertible automata (as the ones of the CP-L-R class, for example), that also have complex dynamics[8].

With CA (as general) the notion of a "final configuration" does not exist, therefore we can not talk about any identical term to the "final state", we know from the classical "Turing machines". That is why the computational process for the CA is always defined as a infinite sequence of generations, generated by a fixed initial configuration.

Definition 13 *(Computational process in the CA)*
Let $G_0 \in \overline{G}$ is a initial configuration in a CA, φ is the global function of the automaton and

$$G_0 \xrightarrow{\varphi} G_1 \xrightarrow{\varphi} \ldots \xrightarrow{\varphi} G_i \xrightarrow{\varphi} \ldots$$

is an infinite sequence of generations, starting with G_0. Then the ordered set $G(G_0) = \{G_0, G_1, \ldots, G_i, \ldots\}$ will be called a computational process in the CA, generated by G_0.

The analysis of the possible computational processes in a given CP-CA is also interesting to see. Before we go to the proof of the last theorem, we will use some basic algebraic notions in order to introduce a definition for "isomorphism of the computational processes".

Definition 14 *(Homomorphism and isomorphism of the computational processes)*
Let $G_0, G'_0 \in \overline{G}$ are initial configurations in CA and φ is the global function of the automaton. What we call homomorphism of the computational process $G(G_0)$ in $G(G'_0)$ is the image $\aleph : G(G_0) \to G(G'_0)$, that complies with the following term:

$$\aleph\left(\varphi(G_i)\right) = \varphi\left(\aleph(G_i)\right),\ \forall G_i \in G(G_0)$$

What we call isomorphism of $G(G_0)$ in $G(G'_0)$ is mutually synonymous homomorphysm. In this case there is an opposite image $\aleph^{-1} : G(G'_0) \to G(G_0)$, that complies with the term:

$$\aleph^{-1}\left(\varphi(G'_i)\right) = \varphi\left(\aleph^{-1}(G'_i)\right),\ \forall G'_i \in G(G'_0)$$

Definition 15 *(Trivial computational process in CP-CA)*
We will say, that $G_0 \in \overline{G}$ defines a trivial computational process $G(G_0)$, if $G_0 : Pop(G_0) = 0$ is the empty configuration in a given $CA \in$ CP-CA.

[7] For comparison: the simplest configuration known by now, that is a GOE in the Conway's game of "Life", gets in a rectangle 9×33 and comprises 226 nonblank cells. The latter was found by a team of researchers at MIT (Roger Banks et. al.) by means of a specially designed for this purpose computer program and after many machine hours—see, for example [1].

[8] Toffoli and Margolus state in [14], that the class of the properly reversible (injective) cellular automata is empty. They mean reversibility on the set of all configurations (not the finite ones, as we do), but this only confirmed our impression, that the structures we have analyzed lie somewhere in the twilight zone.

Theorem 3 *(Isomorphism of the computational processes in reversible CP-CA)*

Let $CA \in$ CP-CA $\cap RCA$, φ is the global function of the automaton and $G_0, G'_0 \in \overline{G} : Pop(G_0), Pop(G'_0) \neq 0$ are non-empty initial configurations in CA. Then isomorphism exists:

$$\Im : G(G_0) \to G(G'_0)$$

i.e. in the reversible CP-CA a singular non-trivial computational process exists that is precise to isomorphism.

Proof. From **Theorem 1** we can immediately conclude the following:

$CA \in$ CP-CA $\cap RCA \Rightarrow \forall W_i$—the reversible chains in CA—are not cyclic.

We demonstrated in it how to find a predecessor $G_{t-1} \in \underline{G}$ to a given configuration $G_t \in \overline{G}$, applying (18) to every reversible chain W_i. We also demonstrated, that if $G_{t-1} \in \overline{G}$, then the so found predecessor is the only finite configuration, that could "give birth" to G_t.

In **Theorem 2** we found a criterion, that lets us to define whether the predecessor G_{t-1}, that was defined in (18), is a finite or infinite configuration, i.e. whether $G_t \in GOE(\overline{G})$ or not.

For better clarity and convenience at the first two proofs we used the $R_i(j)$ while recording (18), but the latter can be entered also as:

$$c^i_{0,j}[t-1] = \bigoplus_{k<j} c^i_{1,k}[t]$$

Using the above more compact record, we can define a function opposite to the global function φ, in the following fashion:

$$\varphi^{-1}(G_t) = \begin{cases} G_{t-1}, & \text{if } \bigoplus\limits_{k} c^i_{1,k}[t] = 0, \ \forall W_i \\ \text{not defined}, & \text{otherwise} \end{cases}$$

where:

$$G_{t-1} = \left\{ c^i_{0,j}[t-1] \in W_i : c^i_{0,j}[t-1] = \bigoplus_{k<j} c^i_{1,k}[t], \ \forall W_i \right\}$$

Let us now create the function γ, that restores the serial (consecutive) number of every generation G_i from a given computation process $G(G_0)$:

$$\gamma\left(G_0, G_i\right) = \begin{cases} 0, & \text{if } G_i = G_0 \\ 1 + \gamma\left(\varphi^{-1}(G_i)\right), & \text{if } G_i \neq G_0 \end{cases}$$

Obviously $\gamma\left(G_0, G_i\right)$ is always defined for $G_i \in G(G_0)$, besides if G_0 is not the empty configuration, then the non-trivial computation process $G(G_0)$ is isomorphous to the set of the natural numbers. As the notion "isomorphism" is transitive, there we get the statement of the Theorem. ■

5 Discussion

The quite natural question rises—why, at the existence of so many non-replicative automata, none of them (as far as we know) has been discovered and analyzed by now? We consider that there are at least several reasons for that:

a) As we have stated many times in the above lines, till now the "parity rules" have not been considered of any particular interest, as they lead only to replications. Many researchers do not even consider it necessary to pay any special attention to this "fact", thinking it is obvious. Such an opinion is so widely spread, as to, for example, Chris Langton states the following in [6]:

> "Any reproductive process going on here resides entirely in the transition rules and not in the configuration itself."

As for the second part of this statement (that the replication is independent of the initial configuration), we can only agree with it, but the conclusion for the transition rules (Fredkin's rules), as we proved in this work, is wrong.

In order to be fair, we have to mention that it seems Langton is one of the many, who share such an "obvious" opinion, but he probably was one of the few who formulated it clearly and emphatically as a "irremediable" fault of the rules in (1).

b) It would be irrelevant, however, to ignore the fact that the definition we give here to the notion of "cellular automaton" is much wider and comprises all discrete homogenous structures, and not only the ones on regular lattice. Obviously, the problem here lies in the construction of the graphs themselves; the reader might try himself to think of a new automaton, different in principle of the ones we analyze here, that meets the definition of CP-CA and to convince himself in that. The construction of such infinite homogenous digraphs is a well difficult task by itself and, obviously, a better algorithm is needed here to automate the process. But, even the generation and the analysis of a particular graph is a rather unpleasant technical work, because in most cases we are not in disposal of any "suitable" representations as the ones in Fig. 7 and have to be content to write down special algorithms.

c) And at last: Why the non-replicative samples of one-dimensional cellular automata presented here were not known by now? The answer here is a simple one: because, even the simplest automaton of the kind—the CP-CYLINDER (6), is nevertheless complex enough to be discovered in any purely experimental way[9].

[9] CP-CYLINDER (6) is described as $\left(2^6, 1\right)$, i.e. (64, 1) linear automaton in the Wolfarm's notation—too big, bearing in mind, that even the most complex analyses of automata of this kind usually do not exceed (4, 1) or (5, 1).

6 Conclusion

In their remarkable book [16] the mathematicians Ian Stewart and Martin Golubitsky provide many spectacular examples of spontaneous destruction or violation of the symmetry of the still and the animated nature.

It happens, for example, that when a spherical milk drop plops in a round cup of milk it generates a completely unexpected form of spill out: it resembles a crown, decorated with 24 spikes, each ending with a small ball!

Some very well known examples might be shattering sometimes, depending on the context we give them: one is the case with the ideal form of the egg that later gives birth to the complex bilateral symmetry of the bird or the reptile. The example of the spiral like shells of some mollusk is analogous, they demonstrate a striking partiality for winding in one of the two possible directions. Despite the bilateral symmetry of the human body the human heart, for example, is on our left side.

May be the most remarkable example, however, remain the double helix of the DNA molecule that, wonder why, occurs only in a right-handed version; its left-handed version has not been found in nature [19].

All these examples from the "Fearful Symmetry: Is God a Geometer?" from Stewart and Golubitsky are very interesting and elegant, but they do not provide us with the smallest notion where this obvious discrepancy could emerge from.

Modeling of ideal gasses using CA, known as "Lattice gases" after Toffoli and Margolus in [14], in the 90-ies turned to be a "substantial scientific business"[10].

Despite all difficulties and obscurity the progress is obvious. In this aspect we hope that the CA model we submit of the CP-violation happens to be another encouraging step in the same direction.

The spontaneous violation of the CP-symmetry could have as a base a much simpler construction than we could have imagined of. The ideas for the discretion of space and time are shared nowadays by physicists and philosophers (see, for example [17], [18]).

The existence of non-replicative, asymmetrical in its essence, behavior of absolutely symmetrical ("balanced") functions in homogenous spaces which at that was generated from only one singular bit (that happens to be a GOE!), can change our concepts for many things.

We would like to complete our work in the same way as [19]. In the context of our presentation this ending acquires a new and unexpected meaning:

> "Finally, the weak force may account for the existence of matter itself in the universe. The physical conservation laws, coupled with considerations of symmetry—essentially an aesthetic principle that guides the formulation of physical theories—strongly suggests that the big bang must have produced equal amounts of matter and antimatter. For every electron there was a positron; for every proton, an antiproton. Since matter and antimatter annihilate each other when they meet, the symmetric conditions in the early universe pose a problem: Why is there any matter—or any antimatter—left in the universe?
>
> It is highly suggestive to suppose that an interaction that violates CP symmetry might account for the asymmetric domination of matter in the universe. That hypothesis admittedly is highly speculative; indeed, it is still too raw even to deserve the name theory. But if the hypothesis survives the test of experiment, the symmetry flaw of the weak force could well turn out to be the crack through which the universe emerged."

[10] Although, as the authors of [14] with their innate sense of humor frankly admit, this is combined still with a "bit of magic".

References

[1] GARDNER, Martin, *Wheels, Life and Other Mathematical Amusements*, W.H.Freeman and Company (1983).

[2] VON NEUMANN, John, *Theory of Self-Reproducing Automata*, edited and completed by A.Burks, University of Illinois Press, Champaign, IL (1966).

[3] WOLFRAM, Stephen, "Statistical Mechanics of Cellular Automata", *Reviews of Modern Physics* **55** (1983), 601–644.

[4] WOLFRAM, Stephen, "Universality and Complexity in Cellular Automata", *Physica D* **10** (1984), 1–35.

[5] FREDKIN, Edward, "Digital Mechanics: An Informational Process Based on Reversible Universal CA", *Physica D* **45** (1990), 254.

[6] LANGTON, Christopher, "Self-Reproduction in Cellular Automata", *Physica D* **10** (1984), 135–144.

[7] MORITA, Kenichi, and Katsutobu IMAI, "Self-Reproduction in Reversible Cellular Space", *International Workshop of Universal Machines and Computation*, Paris (1995) [11].

[8] LANGTON, Christopher—Peter MIKES discussion, *Cellular Automata Mailing List Archieves* (1987) [12].

[9] PETROV, Plamen, "Digital Physics", in preparation.

[10] MOORE, Edward, "Machine Models of Self Reproduction", *Proceedings of Symposia in Applied Mathematics* **14**, American Mathematical Society Society (1962), 17–33.

[11] MYHILL, John, "The Converse to Moore's Garden-of-Eden Theorem", *Proceedings of American Mathematical Society* **14** (1963) 685–686.

[11] See `ftp://kebap.ke.sys.hiroshima-u.ac.jp` and look under `/pub/projects/rca/sr/`

[12] See `ftp://alife.santafe.edu/pub/topics/cas/`

[12] Burks, Arthur, *Essays on Cellular Automata*, University of Illinois Press, Urbana, IL, (1970).

[13] Amoroso, S., and G. Cooper, "The Garden-of-Eden Theorem for Finite Configurations", *Proceedings of American Mathematical Society* **26** (1970) 158–164.

[14] Toffoli, Tomaso, and Norman Margolus, "Invertible Cellular Automata: A Review", *Physica D* **45** (1990), 229–253.

[15] Bennett, Charles, "Notes on the History of Reversible Computation", *IBM Journal of Research and Development* **1** (1988), 17–23.

[16] Stewart, Ian, and Martin Golubitsky *Fearful Symmetry: Is God a Geometer?*, London, Blackwell (1992).

[17] Requardt, Manfred, "Discrete Mathematics and Physics on the Planck-Scale", preprint, Universität Göttingen (1995)[13].

[18] Zurek, Wojciech (editor), *Complexity, Entropy and the Physics of Information*, Addison-Wesley (1989).

[19] Cline, David, "Weak No More", *The Sciencies,* **11–12** (1993), 20–26.

[13] See `http://xxx.lanl.gov` or at `<hep-th@xxx.lanl.gov>` send command in subject line: `get 9504118`.